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On cubic vertex-transitive graphs of given girth

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Abstract

A set of vertices of a graph is distinguishing if the only automorphism that preserves it is the identity. The minimal size of such sets, if they exist, is the distinguishing cost. The distinguishing costs of vertex transitive cubic graphs are well known if they are 1-arc-transitive, or if they have two edge orbits and either have girth 3 or vertex-stabilizers of order 1 or 2.

There are many results about vertex-transitive cubic graphs of girth 4 with two edge orbits, but for larger girth almost nothing is known about the distinguishing costs of such graphs. We prove that cubic vertex-transitive graphs of girth 5 with two edge orbits have distinguishing cost 2, and prove the non-existence of infinite 3-arc-transitive cubic graphs of girth 6.

Keywords: Distinguishing number, distinguishing cost, vertex-transitive cubic graphs, automorphisms, infinite graphs.

Math. Subj. Class.: 05C15, 05C10, 05C25, 05C63, 03E10.

1 Introduction

Cubic vertex-transitive graphs are one of the most studied objects in algebraic graph theory, the first results dating back to the classic works of Foster [14, 15] and Tutte [29]. This topic has remained relevant over time, as evidenced by the works of Coxeter, Frucht, and Powers [9], Djokovic and Miller [10], Goldschmidt [17], Conder and Lorimer [7], Marušič and Scapellato [23], Glover and Marušič [16], Potočnik, Spiga, and Verret [26], and many others (see also [3, 6, 20, 22, 27] for some very recent works on this topic).

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Meanwhile, the *girth* of a graph (shortest cycle length), is a significant characteristic of a graph that appears in numerous well-known problems, results, and formulas. In the class of graphs of bounded girth, one sees that the structure is heavily restricted.

The structure of finite cubic arc-transitive graphs of girth at most 7 and 9 was studied in [13] and [8], respectively, and those of girth 6 were completely determined in [21]. Finite cubic vertex-transitive graphs of girth 3, 4, and 5 were classified by Eiben, Jajcay, and Šparl [12], while the classification of finite cubic vertex-transitive graphs of girth 6 was done by Potočnik and Vidali [28].

Following Albertson and Collins [1], a vertex coloring of a graph Γ is said to be *distinguishing* if the identity is the only automorphism of Γ that preserves the coloring. The smallest number of colors of a distinguishing coloring is the *distinguishing number* $D(\Gamma)$ of Γ. Observe that $D(\Gamma) = 1$ if and only if Γ is asymmetric (has trivial automorphism group), and otherwise $D(\Gamma) \geq 2$. When $D(\Gamma) = 2$ each of the two colors induces a set of vertices which is preserved only by the identity automorphism. We call such sets *distinguishing*, but the term asymmetrizing was also used by Babai [2]. It was proved by Hüning et al, [19] that the only cubic vertex-transitive graphs with $D(\Gamma) > 2$ are $K_{3,3}$, K_4, the cube, and the Petersen graph.

The cardinality of a smallest distinguishing set of a graph Γ is the *2-distinguishing cost*. It was introduced by Boutin [5] in 2008 and denoted $\rho(\Gamma)$. Clearly $0 < \rho(\Gamma) \leq |V(\Gamma)|/2$. Although we cannot talk of the 2-distinguishing cost unless we already know that $D(\Gamma) = 2$, when it is clear from the context, we refer to $\rho(\Gamma)$ as the distinguishing cost, or simply as the cost, without adding that $D(\Gamma) = 2$.

Cubic vertex-transitive graphs can naturally be divided into three categories, depending on the number of edge orbits. Those with one edge orbit are edge-transitive and also arc-transitive, those with three edge orbits have trivial vertex-stabilizers, and are the so-called graphical regular representations (GRRs), and the remaining category are the cubic vertex-transitive graphs with two edge-orbits, which are further divided into the class of *rigid graphs*, whose vertex-stabilizers have order 2, and into the class of *flexible graphs*, which have vertex-stabilizers of order at least 4 (see [20, Corollary 7.2]).

It is easy to observe that GRRs have distinguishing cost equal to 1. In [20] it was proved that cubic arc-transitive graphs (different from $K_{3,3}$, K_4, the cube, and the Petersen graph, which are not 2-distinguishable) have cost at most 5, unless Γ is the infinite 3-valent tree, which has infinite cost. For cubic vertex-transitive graphs with two edge-orbits, the situation is more complex. In [20] it was proved that a cubic vertex-transitive graph which is rigid has distinguishing cost 2, unless it is an infinite ladder, a k-ladder, or a k-Möbius ladder with $k > 3$ (which have distinguishing cost equal to 3). The situation for the flexible cubic vertex-transitive graphs heavily depends on the girth. In [20] it was proved that a flexible cubic vertex-transitive graph of girth 3 has distinguishing cost equal to 2, unless it is a truncation of one of $K_{3,3}$, the Heawood graph or the Tutte-Coxeter graph, in which case it has distinguishing cost equal to 3. For girth 4, a family of cubic vertex-transitive flexible graphs of girth 4 with arbitrarily large distinguishing cost is constructed.

This paper is concerned with the existence of finite and infinite vertex-transitive cubic graphs of certain girths and their distinguishing costs. It continues the investigations of [20] and also solves several open problems posed there. We prove that cubic vertex-transitive graphs of girth 5 with two edge orbits have distinguishing cost equal to 2. In Theorem 4.1 we classify cubic graphs of girth $g \leq 5$ admitting a consistent cycle of length g and such that every edge of Γ is contained in a g-cycle. For $g = 4$ the only example is $K_{3,3}$ and for $g = 5$ the only examples are the Petersen graph and the dodecahedron. In [20], the distinguishing cost of cubic arc-transitive graphs was considered in Section 6. It is known that all such graphs are s-arc regular for $s \leq 5$ or is the infinite trivalent tree. They considered the distinguishing cost for the various values of s, and observed that all known examples are finite. This naturally leads to [20, Question 6.9], which asks if there are any infinite 3-arc-transitive graphs of valency 3 and girth 6. We show there are no such graphs in Corollary 4.7, and give the application of this result to distinguishing cost in Corollary 4.9. Along the way, we also obtain a new proof of Conder and Nedela's result that the only finite connected cubic 3-arc-transitive graphs are the Heawood graph, the Pappus graph, and the Desargues graph.

2 Preliminaries

Let X be a graph. A subgroup $G \leq \mathrm{Aut}\,(X)$ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. In this case the graph X is said to be *G-vertex-transitive*, *G-edge-transitive*, and *G-arc-transitive*, respectively. In the case where $G = \mathrm{Aut}\,(X)$, the symbol G is omitted. An arc-transitive graph is also called *symmetric*. For a positive integer s, an *s-arc* of X is defined as a sequence of vertices $x_0x_1 \ldots x_s$ of X such that x_i is adjacent to x_{i+1} ($i \in \{0, \ldots, s-1\}$) and $x_i \neq x_{i+2}$. A subgroup $G \leq \mathrm{Aut}\,(X)$ is said to be *s-arc-transitive* if it acts transitively on the set of s-arcs of X, and it is said to be *s-regular* if it is s-arc-transitive and the stabilizer of an s-arc in G is trivial. A graph X is said to be *(G, s)-arc-transitive* and *(G, s)-regular* if G is transitive and regular on the set of s-arcs of X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-transitive if the graph is not $(G, s+1)$-arc-transitive. By Weiss [30, 31], for a pentavalent (G, s)-transitive graph, $s \geq 1$, the order of the vertex stabilizer G_v in G is a divisor of $2^{17} \cdot 3^2 \cdot 5$. In addition, the complete classification of vertex-stabilizers can be deduced from his work, as was recently observed by Guo and Feng [18, Theorem 1.1].

Proposition 2.1. [18, Theorem 1.1.] *Let X be a connected pentavalent (G, s)-transitive graph for some $G \leq Aut(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 5$ and one of the following holds:*

(i) For $s = 1$, $G_v \cong \mathbb{Z}_5, D_{10}$ or D_{20};

(ii) For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5;

(iii) For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$.

The following result will be needed for the study of the distinguishing cost of cubic vertex-transitive graphs with girth 5.

Lemma 2.2. *Let X be a connected 5-valent G-arc-transitive graph where G_v is solvable, and let uv be an edge. The only element of G that fixes u, v and all of their neighbors in X is the identity.*

Proof. Follows from the proof of [32, Theorem 4.1]. ■

3 Distinguishing cost of cubic vertex-transitive graphs of girth 5

In this section we will study the distinguishing cost of cubic vertex-transitive graphs of girth 5. Such graphs were classified by Eiben, Jajcay, and Šparl in [12]. Before presenting this classification, we need to introduce some definitions.

Let Λ be a finite k-regular graph and let $D(\Lambda)$ denote the set of its arcs. For a vertex u of Λ, let $\Lambda(u)$ denote the set of neighbors of u in Λ. A *vertex-neighborhood labeling* of Λ is a function $\rho : D(\Lambda) \mapsto \{1, 2, \ldots, k\}$ such that for each $u \in V(\Lambda)$ the restriction of ρ to the set $\{(u, v) : v \in \Lambda(u)\}$ of arcs emanating from u is a bijection. Furthermore, let Y be a graph of order k with $V(Y) = \{v_1, v_2, \ldots, v_k\}$. The generalized truncation $T(\Lambda, \rho; Y)$ of Λ by Y with respect to ρ is the graph with the vertex set $\{(u, v_i) : u \in V(\Lambda), 1 \le i \le k\}$ and edge set $\{(u, v_i)(u, v_j) | v_i v_j \in E(Y)\} \cup \{(u, v_{\rho(u,w)})(w, v_{\rho(w,u)}) | uw \in E(\Lambda)\}$.

Theorem 3.1. [12, Theorem 6.3] *Let Γ be a connected cubic G-vertex-transitive graph of girth 5. Then Γ is isomorphic to the Petersen graph, the dodecahedron, or there exists a connected 5-valent G-arc-transitive graph Λ with the property that the induced action of G_x on the neighbors $\Lambda(x)$ of a vertex x in Λ is isomorphic to C_5 or D_{10} such that Γ is isomorphic to a generalized truncation of Λ by the 5-cycle.*

Remark 3.2. Based on the communication with the authors of [12], it was discovered that the last statement of [12, Theorem 6.3], saying that $\mathrm{Aut}\,(\Gamma) \cong \mathrm{Aut}\,(\Lambda)$ is incorrect, and that the correct statement, which is actually proved in [12] is $\mathrm{Aut}\,(\Gamma) \le \mathrm{Aut}\,(\Lambda)$. The same remark applies to the classifications of cubic vertex-transitive graphs with girth 3 or 4 given in [12].

Corollary 3.3. *Let Γ be a connected cubic G-vertex-transitive graph of girth 5 that is not isomorphic to the Petersen graph or the dodecahedron and $v \in V(\Gamma)$. Then $|G_v| \in \{1, 2, 4\}$.*

Proof. By Theorem 3.1 there exists a connected 5-valent G-arc-transitive graph Λ with the property that the induced action of G_x on $\Lambda(x)$, $x \in V(\Lambda)$, is isomorphic to C_5 or D_{10} such that Γ is isomorphic to a generalized truncation of Λ. Since G_x is not 2-transitive on $\Lambda(x)$ it follows that G does not act transitively on the set of 2-arcs in Λ. By Proposition 2.1 it follows that $|G_x| \in \{5, 10, 20\}$ for any vertex x of Λ. Observe that $|G| = |V(\Lambda)||G_x|$ and $|G| = |V(\Gamma)||G_v|$, where v is a vertex of Γ. Since $|V(\Gamma)| = 5|V(\Lambda)|$ it follows that $|G_v| \in \{1, 2, 4\}$. ∎

Theorem 3.4. *Let Γ be a connected cubic vertex-transitive graph of girth 5 with two edge orbits. Then the distinguishing cost of Γ is 2.*

Proof. Let Γ be a connected cubic vertex-transitive graph of girth 5 with two edge orbits. Let $G = \mathrm{Aut}\,(\Gamma)$. Color the edges of Γ whose orbits induce a perfect matching with the color red, and the other edges with black. Let v_1u_1 be an arbitrary red edge, and let $C_1 = v_1v_2v_3v_4v_5$ and $C_2 = u_1u_2u_3u_4u_5$ be the black cycles of length 5 containing v_1 and u_1 (see Figure 1). First observe that v_1u_1 is the only edge between the cycles C_1 and C_2. This follows from the fact that Γ is a generalized triangulation, and from the definition of generalized triangulations.

Let $S = \{v_2, u_3\}$. We claim that the only element of G that fixes the set S is the identity. Let $g \in \mathrm{Aut}\,(\Gamma)$ fix the set S setwise. Observe that $v_2v_1u_1u_2u_3$ is a $BRBB$

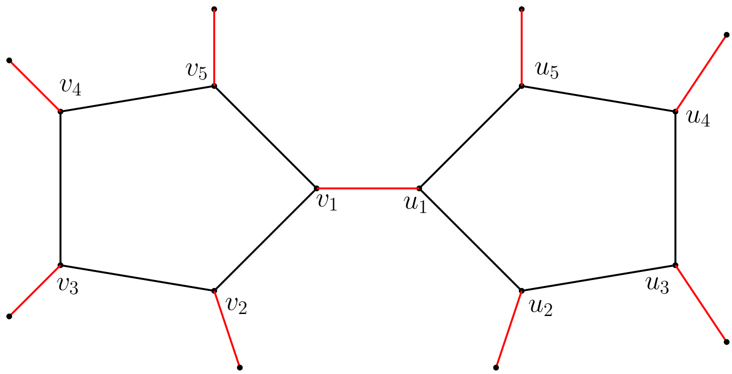

Figure 1: Local structure of Γ around u_1v_1. Black and red edges indicate different edge orbits.

(black-red-black-black) path of length 4 between v_2 and u_3. Since g fixes $\{v_2, u_3\}$ setwise, it follows that g must map this $BRBB$ path of length 4 into another $BRBB$ path of length 4 from v_2 to u_3 or from u_3 to v_2. However, since v_1u_1 is the only red edge between C_1 and C_2 it follows that $v_2v_1u_1u_2u_3$ is the unique $BRBB$ path of length 4 between v_2 and u_3, so all points lying on this 4-path must be fixed, that is, g fixes each of the points v_2, v_1, u_1, u_2, u_3. It follows that g fixes all of the points v_i and u_i, and their neighbors.

Recall that Γ is a generalized truncation of a 5-valent arc-transitive graph Λ. Let $\tilde{G}$ be the subgroup of $\mathrm{Aut}\,(\Lambda)$ induced by the action of G. Note that $G \cong \tilde{G}$ and that $\tilde{G}$ acts 1-arc-transitively on Λ.

Then g corresponds to an automorphism $\tilde{g}$ of Λ, and it follows that $\tilde{g}$ fixes two adjacent vertices of Λ (corresponding to 5-cycles C_1 and C_2) as well as all of their neighbors. By Lemma 2.2 it follows that the only such automorphism of Λ is the identity. We conclude that g is the identity. ■

The next natural step would be to consider cubic vertex-transitive graphs which are flexible with girth larger than 5. The classification of cubic vertex-transitive graphs of girth 6 was obtained by Potočnik and Vidali [28], so one approach would be to try using this classification to determine the distinguishing costs of those graphs, which we leave as an open question for further work.

Problem 3.5. Determine the distinguishing cost of cubic vertex-transitive graphs of girth 6.

Moving to larger girths, the authors are not aware of constructions of cubic flexible vertex-transitive graphs with arbitrary large girth g, which we pose as an open problem.

Problem 3.6. Does there exist a connected flexible cubic vertex-transitive graph of girth g for every positive integer $g \geq 6$?

A cubic vertex-transitive graph that has two edge-orbits is a circular ladder graph or a Möbius ladder graph, or is obtained from a 4-valent arc-transitive graph admitting an arc-transitive cycle decomposition [25, Corollary 13]. By [24], the permutation group induced by the action of the point stabilizer on its neighbors is $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$ or D_4. If the action of the point stabilizer in a 4-valent arc-transitive graph is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, then the obtained cubic vertex-transitive graphs are rigid, and the flexible ones correspond to those obtained from 4-valent arc-transitive graphs with point stabilizer acting locally on neighbours as D_4. It follows that one approach to answer Problem 3.6 is to start with a 4-valent arc-transitive graph of large girth with local action of point stabilizer isomorphic to D_4 admitting the so-called arc-transitive cycle decomposition, and then use the method explained in [25] to construct a cubic flexible graph of large girth.

4 Consistent cycles in cubic graphs

In this section we will study consistent cycles in cubic graphs with small girth. A cycle in a graph is *consistent* if the automorphism group of the graph admits a one-step rotation of this cycle. The main motivation for this section is to answer a question posed in [20, Question 6.9]. We will show in Corollary 4.7 that for $s \geq 3$ there are no infinite cubic s-arc-transitive graphs of girth 6. The authors noticed that similar type results [16, Proposition 3.4] for finite graphs first showed that s-arc-transitive graphs of small girth have consistent cycles. Our proof of Corollary 4.7 uses consistent cycles, which were also central to the arguments of [16, Proposition 3.4] which classifies finite cubic arc-transitive graphs of girth at most five. We will thus first study cubic graphs Γ of small girth with a consistent cycle with the additional condition that every edge of Γ is contained in a girth cycle, an obvious condition that edge-transitive, and hence arc-transitive, graphs also possess.

Theorem 4.1. *Let Γ be a connected cubic graph (finite or infinite) of girth g with a consistent cycle of length g and such that every edge of Γ is contained in a g-cycle. If $g = 4$ then Γ is $K_{3,3}$ or the cube, while if $g = 5$, then Γ is the Petersen graph or the dodecahedron.*

Proof. Let $v_0v_1v_2 \ldots v_{g-1}v_0$ be a consistent g-cycle in Γ. Let $\gamma \in \mathrm{Aut}(\Gamma)$ such that $\gamma(v_i) = v_{i+1}$ with arithmetic in the subscript done modulo g. As Γ is cubic, there exist $u_i \in V(\Gamma)$ with $u_iv_i \in E(\Gamma)$ and no u_i is any v_j, where $i, j \in \mathbb{Z}_g$. We consider various cases depending on the different girths separately.

Case 1: $g = 4$.

Suppose first that some $u_i = u_j$. Note that it cannot be the case that $u_0 = u_1$ or $u_0 = u_3$ as both of those possibilities give a 3-cycle in Γ which has girth 4. Then $u_0 = u_2$. As $\gamma(v_i) = v_{i+1}$ we have that $u_1 = u_3$. If $u_0u_1 \in E(\Gamma)$, then it is easy to see that $\Gamma \cong K_{3,3}$ and the result follows. Otherwise, there is $w_0, w_1 \in V(\Gamma)$ with $u_0w_0, u_1w_1 \in E(\Gamma)$, but $w_0, w_1 \notin \{v_i, u_j : i \in \mathbb{Z}_4, j \in \mathbb{Z}_2\}$. Note that if $w_0 = w_1$, then w_0 is adjacent to some vertex x with $x \notin \{v_i, u_i, w_0\}$. But then w_0x is a bridge of Γ, contradicting the assumption that w_0x is contained in a 4-cycle. So $w_0 \neq w_1$. Let D be a 4-cycle in Γ that contains the edge w_1u_1, and let $\Delta = \Gamma[\{v_i, u_j, w_j : i \in \mathbb{Z}_4, j \in \mathbb{Z}_2\}]$. As every vertex of Δ other than w_0 and w_1 has valency 3, D must contain a path in Δ from u_1 to w_0. However, the distance in Δ from u_1 to w_0 is 4. Thus D cannot exist, a contradiction.

Suppose now that the u_i are all distinct, $i \in \mathbb{Z}_4$. Then $u_0v_0v_1u_1$ is a path on four vertices in Γ, as is $u_3v_3v_0u_0$. Any 4-cycle D in Γ that contains v_0u_0 contains one of the two previously mentioned paths of length 3. So either $u_0v_0v_1u_1u_0$ or $u_3v_3v_0u_0u_3$ is a 4-cycle in Γ. As $\gamma(v_i) = v_{i+1}$, and u_i is the unique neighbor of v_i not on $v_0v_1v_2v_3v_0$, we see $\gamma(u_i) = u_{i+1}$. We conclude in either case that $u_iu_{i+1} \in E(\Gamma)$ for all $i \in \mathbb{Z}_4$. Then Γ is isomorphic to the cube.

Case 2: $g = 5$.

As Γ has girth 5, the u_i, $0 \leq i \leq 4$, are all distinct. If some $u_iu_j \in E(\Gamma)$, $i \neq j$, then, as Γ has girth 5, it must be that u_iu_{i+2} or $u_iu_{i+3} \in E(\Gamma)$. As $\gamma(u_i) = u_{i+1}$, we see that both of these cases occur, and that $u_iu_{i+2}, u_iu_{i+3} \in E(\Gamma)$ for every $i \in \mathbb{Z}_5$. The subgraph induced by the vertices $\{u_i, v_i : i \in \mathbb{Z}_5\}$ is 3-regular, and so all of Γ. By inspection of the edges, this graph is the Petersen graph.

Suppose now that $u_iu_j \notin E(\Gamma)$ for $i \neq j$. Then each u_i is adjacent to two additional vertices, say $w_{i,0}, w_{i,1}$, $i \in \mathbb{Z}_5$. It cannot be the case that the $w_{i,0}$ and $w_{i,1}$ are all distinct, as if that were the case, the edges u_iv_i would not be contained in a 5-cycle. Additionally, $w_{i,0}$ must then be in the same 5-cycle as some of $w_{i\pm 1,0}$ or $w_{i\pm 1,1}$ as, again, the edges u_iv_i would not be contained in a 5-cycle. This means $\{w_{i,j} : i \in \mathbb{Z}_5, j \in \mathbb{Z}_2\}$ consists of 5 distinct vertices $w_0, \ldots, w_4$, and $u_iw_i, u_{i+1}w_i \in E(\Gamma)$. As each w_i has valency 2 in the subgraph of Γ constructed so far, there exist vertices $z_0, \ldots, z_4$, distinct from any u_i, v_i, w_i, such that $w_iz_i \in E(\Gamma)$. Note that if the z_i are not all distinct, then Γ either has girth 4 (if $z_i = z_{i+1}$) or each z_i has valency at least 4 (after applying γ repetitively). The only way in which these edges can be contained in 5-cycles is if $z_iz_{i+1} \in E(\Gamma)$ for all $i \in \mathbb{Z}_4$. The subgraph induced by the vertices $\{u_i, v_i, w_i, z_i : i \in \mathbb{Z}_4\}$ is 3-regular, and so all of Γ. By inspection of the edges, this graph is the dodecahedron. ■

We now consider the case when Γ has girth 6. The proof of the main result is quite a bit longer, and so is broken into two natural cases. In both cases, we show that such a graph is either finite, or has restricted 'local' structure. We also show that in the latter case, the graph cannot have the property that a 3-arc is contained in a 6-cycle.

Lemma 4.2. *Let Γ be a cubic graph of girth 6 (finite or infinite) with a consistent girth cycle and with every edge of Γ contained in a 6-cycle.*
Assume that the subgraph Δ of Γ induced by a consistent cycle and its neighbors has more than 12 edges. Then Γ is either the Heawood graph or contains the graph in Figure 4 as a subgraph. If Γ also has the property that every 3-arc is contained in a 6-cycle, then Γ is the Heawood graph.

Proof. Let $C = v_0v_1 \ldots v_5v_0$ be a consistent cycle in Γ. Let $\gamma \in \mathrm{Aut}(\Gamma)$ such that $\gamma(v_i) = v_{i+1}$ with arithmetic in the subscript done modulo 6. As Γ is cubic and C is a 6-cycle, the vertex v_0 must be adjacent to some other vertex $u_0 \neq v_i$, $i \in \mathbb{Z}_6$. Repetitively applying γ, we see that there are vertices $u_i \in V(\Gamma)$, all distinct from $\{v_i : i \in \mathbb{Z}_6\}$, and $v_iu_i \in E(\Gamma)$ for every $i \in \mathbb{Z}_6$. This subgraph of Γ is shown in Figure 2, and will be called the base graph.

Note that the vertex sets of the base graph and Δ are the same. As we are assuming Δ has more than 12 edges, some of the u_i, $i \in \mathbb{Z}_6$, are adjacent to each other. As Γ has girth 6, the only possibilities are that $u_iu_{i+3} \in E(\Gamma)$ for some $i \in \mathbb{Z}_6$. Applying powers of γ to this edge, we see that $u_iu_{i+3} \in E(\Gamma)$ for every $i \in \mathbb{Z}_6$. We have now determined 15 edges in Γ with v_i of valency 3 and u_i of valence 2 in the subgraph of Γ constructed so far, and each such edge is contained in a 6-cycle. Thus each u_i is adjacent to some other vertex of Γ, and this other vertex is neither a v_i or a u_i. We let $w_0 \in V(\Gamma)$ with $u_0w_0 \in E(\Gamma)$. We will consider two cases, depending upon whether or not w_0 is adjacent to more than one u_i, $i \in \mathbb{Z}_6$.

If w_0 is adjacent to more than one u_i, then, as Γ has girth 6, w_0 cannot be adjacent to u_1, u_3, or u_5. Applying γ to the edges and vertices we have identified so far, we see $u_1w_1 \in E(\Gamma)$ for some $w_1 \in V(\Gamma)$, $v_i \neq w_i \neq u_1$, $i \in \mathbb{Z}_6$, and $w_1 \neq w_0$ (as Γ has girth

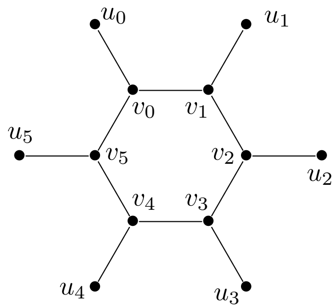

Figure 2: The Base Graph.

6). If w_0 is adjacent to u_2, then w_1 is adjacent to u_3. Applying γ to the edges and vertices we have now found, we see that w_0 is adjacent to u_4 and w_1 is adjacent to u_5. Every vertex is now cubic, and so the graph has been constructed. See the left-hand side of Figure 3. We observe that the given graph is the Heawood graph - the right-hand side of Figure 3 with the labeling used here is a usual drawing of the Heawood graph (see [11, Figure 4.4]). The case where w_0 is adjacent to u_4 is the reflection of the case where w_0 is adjacent to u_2, and so also results in the Heawood graph.

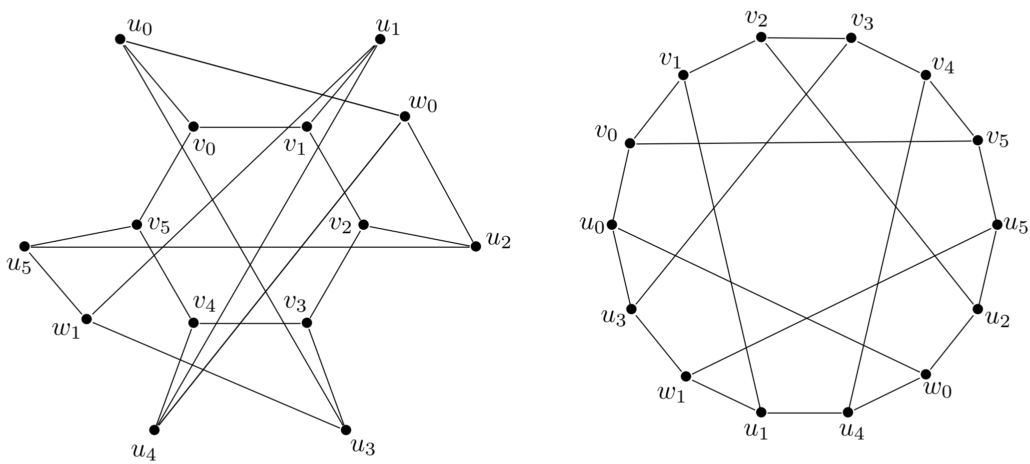

Figure 3: The Heawood graph.

If w_0 is adjacent to exactly one u_i, then, repeatedly applying γ, we see that each u_i is adjacent to a vertex w_i and $w_i \notin \{v_i, u_i : i \in \mathbb{Z}_6\}$. Relabeling if necessary, there are vertices $w_0, \ldots, w_5$, none of which are in $\{v_i, u_i : i \in \mathbb{Z}_6\}$, and $u_i w_i \in E(\Gamma)$. We have now identified 18 vertices of Γ. Each v_i has valency 3, each u_i has valency 3, and each w_i has valency 1. All edges except for the edges $u_i w_i$, $i \in \mathbb{Z}_6$ are contained in cycles of length 6. This is the graph in Figure 4, which for convenience we will call Ω.

Consider the 3-arc $A = u_0 v_0 v_1 u_1$ in Γ. In order for A to be contained in a 6-cycle C'

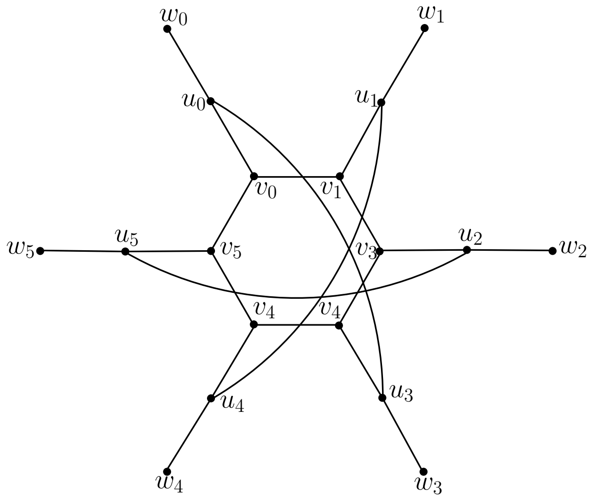

Figure 4: A subgraph of Γ.

of Γ, some neighbor of u_0 not on A must be adjacent to some neighbor of u_1 not on A in Γ. The neighbors of u_0 and u_1 not on A in Γ are all contained in Ω, and are the vertices w_0, w_1, u_3, and u_4. In Ω, these vertices form an independent set, so C' must contain an edge not in Ω, namely, w_0w_1. Applying γ to the edge w_0w_1, we see that $w_iw_{i+1} \in E(\Gamma)$, and we obtain the cubic graph Λ in Figure 5.

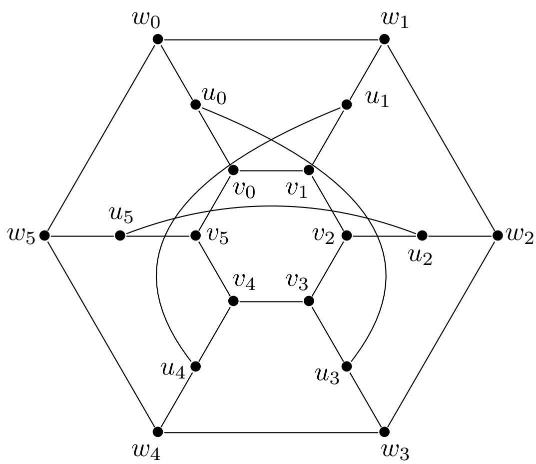

Figure 5: A graph of order 18.

Consider the 3-arc $v_5u_5u_2w_2$ in Λ. In order to be contained in a cycle in Λ, the neighbors v_0, v_4 of v_5 and the neighbors w_1 and w_3 of w_2 in Λ must have a common neighbor. However, $\{v_0, v_4, w_1, w_3\}$ is an independent set in Λ. The result follows. ■

The graph in Figure 5 is reminiscent of the Pappus graph, but according to Magma [4] has automorphism group of order 24, and was constructed to contain a consistent cycle. Thus it is an example of a cubic graph of girth 6 admitting a consistent cycle with every edge being contained in a girth cycle but which is not vertex-transitive.

Lemma 4.3. *Let* Γ *be a cubic graph of girth* 6 *(finite or infinite) that satisfies the following conditions:*

1. Γ *has a consistent girth cycle,*
2. *every* 3*-arc is contained in a* 6*-cycle, and*
3. *the subgraph* Δ *of* Γ *induced by a consistent cycle and its neighbors has* 12 *edges.*

Then Γ *is either the Pappus graph or the Desargues graph.*

Proof. Let $D = v_0v_1v_2v_3v_4v_5v_0$ be a consistent 6-cycle in Γ with $\gamma \in \mathrm{Aut}\,(\Gamma)$ such that $\gamma(v_i) = v_{i+1}$. For each v_i there is $u_i \in \Gamma$ such that $u_iv_i \in E(\Gamma)$, $i \in \mathbb{Z}_6$. Let $U = \{u_i : i \in \mathbb{Z}_6\}$, $V = \{v_i : i \in \mathbb{Z}_6\}$, and $W = \{w \in V(\Gamma) : wu_j \in E(\Gamma) \text{ and } w \neq v_k, j, k \in \mathbb{Z}_6\}$. By hypothesis, the subgraph Δ of Γ induced by $U \cup V$ is the base graph in Figure 2.

By hypothesis, the 3-arc $u_0v_0v_1v_2$ must be contained in a 6-cycle C. We must also have a neighbor of u_0 which is not v_0 on C, which we will call w_0. As Δ has 12 edges we must have that $w_0 \notin V(\Delta)$.

First observe that W has at most 12 vertices, as there are 12 edges between U and W. Consider the action of γ on W. Suppose that γ has an orbit of length k on W, call it W_0. Then each of the k vertices of W_0 has the same number of neighbors in U. The number of edges between U and W_0 is equal either to 6 or 12, as each vertex of U has either one or two neighbours in W_0. We conclude that k must be divisor of 12, that is $k \in \{1, 2, 3, 4, 6, 12\}$. Observe also that γ has at most two orbits on W, as there are at least 6 edges between U and an orbit of γ on W, and there are 12 edges in total between U and W.

If $w \in W$ and $\gamma(w) = w$, then as γ is a 6-cycle on U, we see that w has valency at least 6, a contradiction. Hence no element of W is fixed by γ, that is γ has no orbit of size 1 on W.

Suppose first that γ has an orbit W_0 of length 4 on W. We may assume that $W_0 = \{w_0, w_1, w_2, w_3\}$, $\gamma(w_i) = w_{i+1}$, and $w_0u_0 \in E(\Gamma)$. By the action of γ we conclude that both w_0 and w_2 are adjacent to u_0, u_2, and u_4. Thus $w_0u_0w_2u_2w_0$ is a 4-cycle in Γ, which is impossible as Γ has girth 6. So γ has no orbit of length 4 on W.

Suppose that γ has an orbit of length 12 on W. Then we can assume that $W = \{w_0, \ldots, w_{11}\}$, $\gamma(w_i) = w_{i+1}$, and $u_0w_0 \in E(\Gamma)$. The action of γ implies that u_i is adjacent with w_i and w_{i+6}, for each $i \in \{0, \ldots, 5\}$. The cycle C containing the 3-arc $u_0v_0v_1v_2$ must contain the edge v_2u_2 or v_2v_3. If v_2u_2 is contained in C, then u_2 has to be adjacent with one of w_0 or w_6, which is impossible. Similarly, if C contains v_2v_3 then v_3 has to be adjacent with one of w_0 or w_6, which is also impossible. This shows that γ has no orbit of size 12 on W, hence the only remaining possible sizes of orbits of γ on W are 2, 3, and 6.

Suppose now that γ has an orbit of length 2 on W, and suppose that $\{x_0, x_1\}$ is this orbit. Without loss of generality we may assume that x_0u_0 is an edge of Γ. Then by the action of γ we have that x_0 is adjacent with u_0, u_2 and u_4, while x_1 is adjacent with u_1, u_3 and u_5. Then γ has another orbit on W, call it W_1.

If W_1 has size 2, then we get another vertex w_0 which is adjacent to u_0, u_2, u_4. However, this would give us a 4-cycle $x_0u_0w_0u_2x_0$ in Γ, which is impossible as Γ has girth 6. If W_1 has size 3, then we can assume that $W_0 = \{w_0, w_1, w_2\}$, $\gamma(w_i) = w_{i+1}$ and $u_0w_0 \in E(\Gamma)$. By the action of γ we get that w_i is adjacent with u_i and u_{i+3}, for $i \in \{0, 1, 2\}$. Consider the 3-arc $u_0v_0v_1u_1$. The only way this 3-arc lies on a 6-cycle is to have an edge between w_0 and w_1. However, we would obtain a 3-cycle $w_0w_1w_2w_0$ in Γ which is impossible.

We conclude that W_1 has size 6, so we can assume that $W_1 = \{w_0, w_1, w_2, w_3, w_4, w_5\}$, $\gamma(w_i) = w_{i+1}$, and u_0w_0 is an edge of Γ. By the action of γ we conclude that u_iw_i is an edge for each $i \in \mathbb{Z}_6$. Consider the 3-arc $u_0v_0v_1u_1$. The only way this 3-arc lies on a 6-cycle is to have an edge between w_0 and w_1. By the action of γ we obtain that w_iw_{i+1} is an edge for each $i \in \mathbb{Z}_6$. Observe that Γ has no more vertices, as each vertex has 3 neighbors in the graph constructed so far, hence Γ is determined. It is the graph on the left side of Figure 6, and on the right side is a standard drawing of the Desargues graph. We see that Γ is isomorphic to the Desaurges graph.

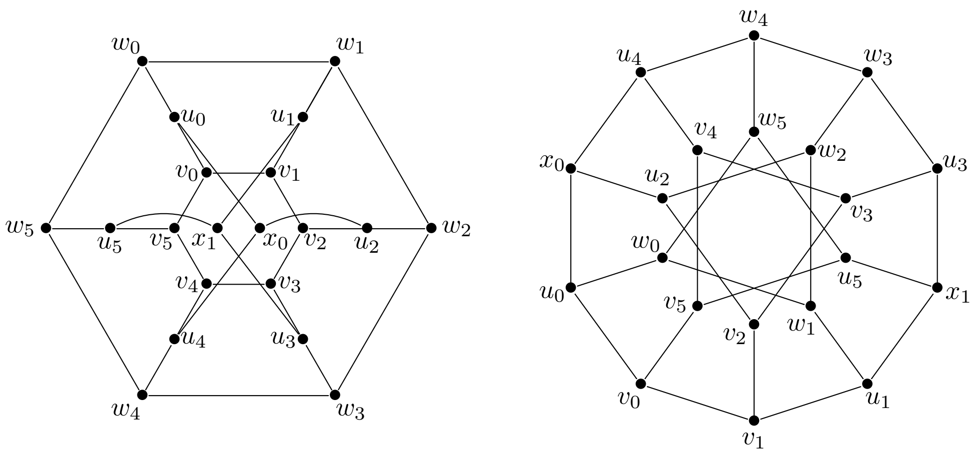

Figure 6: The Desargues graph.

Suppose now that γ has an orbit W_0 of size 3 on W and no orbit of size 2. Then we may assume that $W_0 = \{w_0, w_1, w_2\}$, $\gamma(w_i) = w_{i+1}$, and w_i is adjacent with u_i and u_{i+3}, for each $i \in \{0, 1, 2\}$. If γ has another orbit of size 3 on W, then there is another vertex w that is adjacent to both of u_0 and u_3. However, this would produce a 4-cycle $w_0u_0wu_3w_0$, which is impossible, since Γ has girth 6. We conclude that γ has another orbit of size 6 on W. We may assume that this orbit is $\{x_0, \dots, x_5\}$, $\gamma(x_i) = x_{i+1}$, and x_i is adjacent with u_i, for each $i \in \mathbb{Z}_6$. It is easy to see that the 3-arc $u_0v_0v_1v_2$ does not lie on any 6-cycle, a contradiction. We conclude that γ has no orbit of size 3 on W.

The remaining case to consider is when every orbit of γ on W has size 6. Suppose that there are 2 such orbits $W_0 = \{w_0, \dots, w_5\}$ and $W_1 = \{x_0, \dots, x_5\}$. We can also assume that u_i is adjacent with w_i and x_i for each $i \in \mathbb{Z}_6$. It is easy to see that the 3-arc $u_0v_0v_1v_2$ does not lie on any 6-cycle, a contradiction.

We conclude that $|W| = 6$ and there is just one orbit of γ on W. Hence we may assume that $W = \{w_0, \ldots, w_5\}$ and that u_i is adjacent with w_i for each $i \in \mathbb{Z}_6$. It is now easy to see that the only way that the 3-arc $u_0v_0v_1v_2$ lies on 6-cycle is to have that w_0u_2 or w_2u_0 is an edge of Γ. It follows that w_iu_{i+2} ($i \in \mathbb{Z}_6$) are edges of Γ or w_iu_{i-2} ($i \in \mathbb{Z}_6$) are edges of Γ. As the second option can be obtained from the first by reflection (replacing index i in each vertex with $-i$), without loss of generality we will assume that w_iu_{i+2} ($i \in \mathbb{Z}_6$) are edges of Γ.

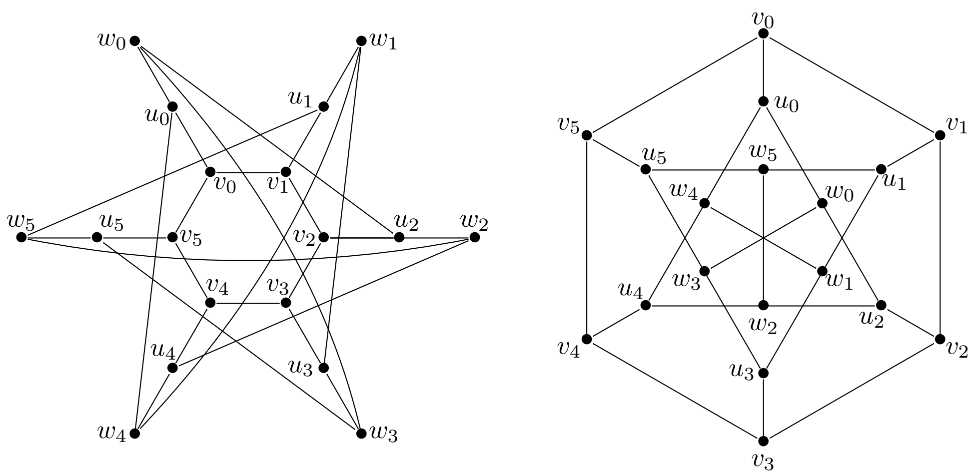

Figure 7: The Pappus graph.

Since the 3-arc $u_0v_0v_1u_1$ lies on a 6-cycle, it follows that there is an edge between some of the neighbors of u_0 and u_1 different from v_0 and v_1. It follows that there is at least one edge between vertices in W. Since γ induces a cycle of length 6 on W, applying γ, it follows that each vertex of W has a neighbor in W. It is now easy to see that the only possibility is that w_iw_{i+3} is an edge of Γ, as otherwise each vertex of W would have two neighbors in W and two neighbors in U, contradicting the assumption that Γ is cubic. Since Γ is connected, it follows that Γ is of order 18, and it can be seen that it is isomorphic to the Pappus graph (see left hand side of Figure 7). This concludes the proof. ∎

The next result is a combination of Lemmas 4.2 and 4.3

Theorem 4.4. *Let Γ be a cubic graph of girth 6 (finite or infinite) that contains a consistent girth cycle and such that every 3-arc of Γ is contained in a 6-cycle. Then Γ is the Heawood graph, the Pappus graph, or the Desaurges graph.*

We next turn our attention to which cubic s-arc-transitive graphs contain consistent cycles.

Lemma 4.5. *Let $s \geq 3$. Every cubic s-arc-transitive graph Γ of girth $s + 2$ contains a consistent cycle.*

Proof. Let Γ be an s-arc-transitive graph of girth $s + 2$, and let $A = v_0, v_1, \ldots, v_s$ be an s-arc in Γ. As Γ is s-arc-transitive, has girth $s + 2$, and an $(s + 2)$-cycle contains an s-arc, we see that A is contained in an $(s+2)$-cycle $C = v_0 v_1 v_2 \ldots v_{s+1} v_0$. Then there exists $\gamma \in \mathrm{Aut}\,(\Gamma)$ such that $\gamma(A) = v_1, v_2, \ldots, v_s, v_{s+1}$. Hence $\gamma(v_i) = v_{i+1}$ for $0 \leq i \leq s$. If $\gamma(v_{s+1}) = v_0$, then C is a consistent cycle in Γ. Otherwise, $\gamma(v_{s+1}) = x \notin V(C)$. Then $\gamma(v_{s+1}v_0) = xv_1$, and $v_{s+1}xv_1v_0v_{s+1}$ is a 4-cycle in Γ, a contradiction with the girth of Γ being equal to $s+2 \geq 5$. Hence Γ contains a consistent cycle. ■

Lemma 4.6. *Every* 3*-arc-transitive graph of girth* 6 *has a consistent* 6*-cycle.*

Proof. Suppose that Γ is a cubic 3-arc-transitive graph of girth 6 without a consistent 6-cycle.

Claim: Given any 6-cycle of Γ, any two antipodal points of the 6-cycle are endpoints of a path of length 3 sharing no edge with the 6-cycle.

Let $C_0 = v_0v_1v_2v_3v_4v_5v_0$ be a cycle of length 6 in Γ. As Γ is 3-arc-transitive, there exists an automorphism α mapping the 3-arc $v_0v_1v_2v_3$ to $v_1v_2v_3v_4$. If $\alpha(v_4) = v_5$, then either $\alpha(v_5) = v_0$ and C_0 is a consistent 6-cycle in Γ, or $\alpha(v_5) = z \notin V(C_0)$. But then $\alpha(v_4v_5v_0) = v_5zv_1$ and $v_5zv_1v_0v_5$ is a 4-cycle in Γ, a contradiction showing that $\alpha(v_4) \neq v_5$.

Suppose now that $\alpha(v_4) = x \neq v_5$. Then $\alpha(\{v_3, v_4\}) = \{v_4, x\} \in E(\Gamma)$. Let $w = \alpha(v_5)$. Note that as Γ has girth 6, $w \notin V(C_0)$. Then $\alpha(\{v_5, v_0\}) = \{w, v_1\} \in E(\Gamma)$ and v_4xwv_1 is the claimed path of length 3, concluding the proof of the claim.

Denote by u_i the neighbour of v_i outside of C_0, for $i \in \mathbb{Z}_6$, and so we have the base graph in Figure 2 as a subgraph of Γ. By the above claim, it follows that u_0u_3, u_1u_4 and u_2u_5 are edges in Γ. Consider now the 6-cycle $C_1 = v_0v_1v_2u_2u_5v_5v_0$, and observe that v_0, u_2 is a pair of antipodal points of C_1. By the Claim there exists a path of length 3 between them sharing no edge with C_1. It is clear that u_0 must be contained in this path, as it is the only neighbour of v_0 outside of C_1. Let x be the remaining vertex of this path. Since Γ is cubic of girth 6 it follows that x is different from all v_i and u_i. Hence x is adjacent with u_0 and u_2.

Consider now the 6-cycle $C_2 = v_0v_1u_1u_4v_4v_5v_0$ and observe that v_0, u_4 is a pair of antipodal points of C_2. Hence there is a path of length 3 between v_0 and u_4 sharing no edge with C_2. Observe that v_0u_0 is one edge of this path. Also, u_3 cannot be contained on this path, since otherwise we would get the 4-cycle $v_3v_4u_4u_3v_3$. It follows that $v_0u_0xu_4$ must be the path, that is x is adjacent with u_0, u_2 and u_4.

Considering 6-cycles $v_1v_2v_3u_3u_0v_0v_1$ and $v_3v_4v_5u_5u_2v_2v_3$, it follows that there is another vertex y adjacent with u_1, u_3 and u_5. Since the component D of Γ that contains v_1 is connected and cubic, it follows that D is of order 14. From the construction of D it follows that D is the graph on the left hand side of Figure 3, and so is isomorphic to the Heawood graph. But the Heawood graph is 4-arc-regular, and, as is well known has a consistent 6-cycle (this also follows by Lemma 4.5). ■

We now prove the main result of this section, and show there are no infinite s-arc-transitive graphs of girth 6 for $s \geq 3$.

Corollary 4.7. *Let $s \geq 3$. There are no infinite cubic s-arc-transitive graphs of girth* 6*.*

Proof. As an s-arc-transitive graph is 3-arc-transitive, it suffices to show that there are no infinite cubic 3-arc-transitive graphs of girth 6. If such a graph Γ exists, it has a consistent cycle $C = v_0v_1v_2v_3v_4v_5v_0$ by Lemma 4.6. Let $xy \in E(\Gamma)$. As Γ is cubic, xy is contained in some 3-arc A of Γ. As Γ is 3-arc-transitive, there exists $\delta \in \mathrm{Aut}(\Gamma)$ such that $\delta(v_0v_1v_2v_3) = A$. Then xy is contained in the 6-cycle $\delta(C)$. We conclude that every edge of Γ is contained in a girth cycle of Γ. The result now follows by Theorem 4.4. ■

Corollary 4.8. *Let $s \geq 3$. The only connected cubic s-arc-transitive graphs of girth 6 are the Heawood graph, the Desargues graph, and the Pappus graph.*

We now give applications of the above results for the distinguishing cost for cubic arc-transitive graphs of girth 6. Our first result improves [20, Lemma 6.8].

Corollary 4.9. *Let Γ be a connected cubic arc-transitive graph of girth 6. Then Γ is at most 4-arc-regular. Also,*

1. *if Γ is 1-arc-regular, then $\rho(\Gamma) = 2$,*
2. *if Γ is 2-arc-regular, then $\rho(\Gamma) \leq 3$,*
3. *if Γ is 3-arc-regular, then Γ is the Desaurges graph or the Pappus graph, and $\rho(\Gamma) = 3$,*
4. *if Γ is 4-arc-regular, then Γ is the Heawood graph and $\rho(\Gamma) = 5$.*

Proof. All of the information in the result for $s = 1$ and 2 and comes from [20, Lemma 6.8]. Except for $\rho(\Gamma)$ in (3) and (4), the result follows by Corollary 4.8. The distinguishing cost of the Desaurgue, Pappus, and Heawood graphs are easily verified by Magma. ■

The next result lowers the upper bound on the distinguishing cost given in [20, Theorem 6.1] from 5 to 4 by additionally excluding the Heawood graph. See [20, Section 3] for the definition of distinguishing density.

Corollary 4.10. *Let Γ be a cubic arc-transitive graph that is not K_4, $K_{3,3}$, the cube, the Petersen graph, or the Heawood graph. If Γ has finite girth, then $\rho(\Gamma) \leq 4$, and otherwise it is the infinite cubic tree T_3, which has finite distinguishing cost and distinguishing density 0.*

Proof. If Γ does not have finite girth, then the result follows by [20, Theorem 6.1]. If Γ has finite girth g, then if $g \geq 7$ we have $\rho(\Gamma) \leq 4$ by [20, Lemma 6.7]. If $g \leq 5$, then by [20, Theorem 6.2] all such graphs have been excluded. If $g = 6$, then by Corollary 4.9 the only such graph with $\rho(\Gamma) > 4$ is the Heawood graph, which has also been excluded. ■

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